

# SOME ONE-SIDED ESTIMATES FOR OSCILLATORY SINGULAR INTEGRALS

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**ABSTRACT.** The purpose of this paper is to establish some one-sided estimates for oscillatory singular integrals. The boundedness of certain oscillatory singular integral on weighted Hardy spaces  $H_+^1(w)$  is proved. It is here also show that the  $H_+^1(w)$  theory of oscillatory singular integrals above cannot be extended to the case of  $H_+^q(w)$  when  $0 < q < 1$  and  $w \in A_p^+$ , a wider weight class than the classical Muckenhoupt class. Furthermore, a criterion on the weighted  $L^p$ -boundedness of the oscillatory singular integral is given.

## 1. INTRODUCTION AND MAIN RESULTS

The study of one-sided operators was motivated not only as the generalization of the theory of two-sided ones but also by the demand in ergodic theory [4], [7]. The well-known Riemann-Liouville fractional integral can be viewed as the one-sided version of Riesz potential [27]. In [40], Sawyer studied the weighted theory of one-sided maximal Hardy-Littlewood operators in depth for the first time. Since then, numerous papers have appeared, among which we choose to refer to [2], [3], [11], [24], [25] and [37] about one-sided operators, [1], [10], [30], [34], [35], and [38] about one-sided spaces, respectively. Interestingly, lots of results show that for a class of smaller operators (one-sided operators) and a class of wider weights (one-sided weights), many famous results in harmonic analysis still hold.

Besides the Hardy-Littlewood maximal operators and Calderón-Zygmund singular integral operators, oscillatory integral operators have played an important role in harmonic analysis from its outset; three chapters are devoted to them in the celebrated Stein's book [42]. Many important operators in harmonic analysis are some versions of oscillatory integrals, such as Fourier transform, Bochner-Riesz means, Radon transform in CT technology and so on. For a more complete account on oscillatory integrals in classical harmonic analysis, we would like to refer the interested reader to [13], [20], [21], [22], [23], [32], [33], [39] and references therein. In more recent times, the operators fashioned from oscillatory integrals, such as pseudo-differential

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operator in PDE become another motivation to study them. Based on the estimates of some kinds of oscillatory integrals, one can establish the well-posedness theory of a class of dispersive equations, for some of these works, we refer to [5], [17] and [18].

Inspired by theory of oscillatory singular operators and one-sided operators, the authors of this paper defined the one-sided oscillatory integral operator in [11] (see also [12]), which is just the object of this paper. We first recall its definitions as

$$T^+ f(x) = \lim_{\varepsilon \rightarrow 0^+} \int_{x+\varepsilon}^{\infty} e^{iP(x,y)} K(x-y) f(y) dy = \text{p.v.} \int_x^{\infty} e^{iP(x,y)} K(x-y) f(y) dy$$

and

$$T^- f(x) = \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{x-\varepsilon} e^{iP(x,y)} K(x-y) f(y) dy = \text{p.v.} \int_{-\infty}^x e^{iP(x,y)} K(x-y) f(y) dy,$$

where  $P(x, y)$  is a real polynomial defined on  $\mathbb{R} \times \mathbb{R}$ , and  $K$  is a one-sided Calderón-Zygmund kernel with support in  $\mathbb{R}^- = (-\infty, 0)$  and  $\mathbb{R}^+ = (0, +\infty)$ , respectively. We recall that a function  $K \in L_{loc}^1\{\mathbb{R}/\{0\}\}$  is a Calderón-Zygmund kernel, if there exists a finite constant  $C$  such that the following properties are satisfied:

(a)

$$\left| \int_{a < |x| < b} K(x) dx \right| \leq C, \quad 0 < a < b$$

holds for all  $a$  and  $b$ , and there exists  $\lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon < |x| < 1} K(x) dx$ ,

(b)

$$|K(x)| \leq C/|x|, \quad x \neq 0,$$

(c)

$$|K(x-y) - K(x)| \leq C|y|/|x|^2$$

for all  $x$  and  $y$  with  $|x| > 2|y| > 0$ .

A Calderón-Zygmund kernel with support in  $(-\infty, 0)$  (or in  $(0, \infty)$ ) will be called the one-sided Calderón-Zygmund kernel. In [2], Aimar, Forzani and Martín-Reyes give an example of such kernel

$$K(x) = \frac{\sin(\log |x|)}{(x \log |x|)} \chi_{(-\infty, 0)}(x),$$

where  $\chi_E$  denotes the characteristic function on a set  $E$ .

In order to give the main results of our paper, some definitions and propositions for one-sided weights are needed. Let  $f(x)$  be a measurable function defined on  $\mathbb{R}$ . The one-sided Hardy-Littlewood maximal functions  $M^+ f(x)$  and  $M^- f(x)$  are defined by

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)| dy$$

and

$$M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^x |f(y)| dy,$$

which arose in the ergodic maximal function, see [40].

As usual, a weight  $w(x)$  is a measurable and non-negative function. If  $O \subset \mathbb{R}$  is a Lebesgue measurable set, we denote its  $w$ -measure by  $w(O) = \int_O w(t) dt$ . A function

$f(x)$  belongs to  $L^p(w)$  ( $0 < p \leq \infty$ ), if  $\|f\|_{L^p(w)} = (\int_{\mathbb{R}} |f(x)|^p w(x) dx)^{1/p} < \infty$ . If  $w(x) = 1$ , we denote  $\|f\|_{L^p(w)}$  simply by  $\|f\|_{L^p(\mathbb{R})}$ . The classical Dunford-Schwartz ergodic theorem (see [7]) can be considered as the first result about weights (one-sided weights) for  $M^+$  and  $M^-$ . A weight  $w(x)$  belongs to the class  $A_p^+$ ,  $A_p^-$  (one-sided  $A_p$  weights) defined by Sawyer [40], if they satisfy the following conditions:

$$A_p^+(w) =: \sup_{a < b < c} \frac{1}{(c-a)^p} \int_a^b w(x) dx \left( \int_b^c w(x)^{1-p'} dx \right)^{p-1} < \infty,$$

$$A_p^-(w) =: \sup_{a < b < c} \frac{1}{(c-a)^p} \int_b^c w(x) dx \left( \int_a^b w(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where  $1 < p < \infty$ . When  $p = 1$  and  $p = \infty$ ,

$$(1.1) \quad A_1^+ : M^- w \leq Cw, \quad A_1^- : M^+ w \leq Cw,$$

for some constant  $C$  and

$$A_\infty^+ = \cup_{p>1} A_p^+, \quad A_\infty^- = \cup_{p>1} A_p^-.$$

The smallest constant  $C$  for which (1.1) is satisfied will be denoted by  $A_1^+(w)$  and  $A_1^-(w)$ .  $A_p^+(w)$  (or  $A_p^-(w)$ ),  $p \geq 1$ , will be called the  $A_p^+$  (or  $A_p^-$ ) constant of  $w$ . Here  $A_p$  ( $1 \leq p \leq \infty$ ) denotes the Muckenhoupt class [28]. This class consists of weight functions  $w$  for which

$$A_p : \sup_I \left( \frac{1}{|I|} \int_I w(x) dx \right) \left( \frac{1}{|I|} \int_I w(x)^{1-p'} dx \right)^{p-1} < \infty, \quad 1 < p < \infty,$$

$$A_1 : Mw \leq Cw,$$

and

$$A_\infty = \cup_{p>1} A_p,$$

where the supremum is taken over all intervals  $I \subset \mathbb{R}$ ,  $1/p + 1/p' = 1$  and  $M$  is the classical Hardy-Littlewood maximal function  $Mf(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x+h} |f(y)| dy$ . Throughout this paper, the letter  $C$  is used for various constants, and may change from one occurrence to another.

Given  $w(x) \in A_p^+$ ,  $1 \leq p < \infty$ , one can define  $x_{-\infty}$  and  $x_{+\infty}$  with  $-\infty \leq x_{-\infty} \leq x_{+\infty} \leq \infty$  (see [36]), such that

- (a)  $w(x) = 0$ ,  $x \in (-\infty, x_{-\infty})$ ;
- (b)  $w(x) > 0$ ,  $x \in (x_{-\infty}, +\infty)$ .

In [40], Sawyer obtained the characterizations of the weighted inequalities for  $M^+$  and  $M^-$ , respectively. He proved that  $M^+$  was bounded on  $L^p(w)$  if and only if  $w \in A_p^+$  with  $1 < p < \infty$ . For  $p = 1$ , Sawyer also showed the weak weighted estimate for  $M^+$  holds if and only if  $w \in A_1^+$ . Let us remark here that similar results can be obtained for the left-hand-side operator (for example,  $M^-$ ) by changing the condition  $A_p^+$  by  $A_p^-$ . Together with the characterizations of the weighted inequalities for  $M^+$  and  $M^-$ , Sawyer obtained some properties of the classes  $A_p^+$  and  $A_p^-$ :

- (a) If  $w \in A_1^+$ , then  $w^{1+\varepsilon} \in A_1^+$  for some  $\varepsilon > 0$ .
- (b) If  $1 \leq p < \infty$ , then  $A_p = A_p^+ \cap A_p^-$ ,  $A_p \subset A_p^+$ ,  $A_p \subset A_p^-$ .
- (c)  $A_p^+ \subset A_r^+$ ,  $A_p^- \subset A_r^-$  if  $1 \leq p \leq r$ .

We conclude from (b) that  $A_p^+$  class is a wider class than  $A_p$  class. Take  $e^x$  for example,  $e^x \notin A_1$ , but  $e^x \in A_1^+$ . For more information about weights for one-sided operators, we refer the reader to a survey article of Martín-Reyes, Ortega and Torre [25].

Since  $T^+$  is not a Calderón-Zygmund operator, we cannot directly use the Calderón-Zygmund theory to study this boundedness. Also, since  $T^+$  is not a convolution operator, we cannot use Fourier transform method either. The study of the boundedness of  $T^+$  centers on three main questions which we shall describe below:

When  $p = 1$ , both  $M^+$  and the one-sided Calderón-Zygmund singular integral operator  $\tilde{T}^+$  which defined by

$$\tilde{T}^+ f(x) = \text{p.v.} \int_x^\infty K(x-y)f(y)dy = \lim_{\varepsilon \rightarrow 0^+} \int_{x+\varepsilon}^\infty K(x-y)f(y)dy$$

are bounded from  $L^1(w)$  to  $L^{1,\infty}(w)$  with  $w \in A_1^+$ , where  $L^{1,\infty}(w)$  denotes the weak  $L^1(w)$  space with the seminorm  $\|f\|_{L^{1,\infty}(w)} := \sup_{\lambda>0} \lambda w(\{x \in \mathbb{R} : |f(x)| > \lambda\})$ , see [25]. Therefore, one naturally wants to know

**Question 1** Is  $T^+$  bounded from  $L^1(w)$  to  $L^{1,\infty}(w)$  with  $w \in A_1^+$ ?

We answer Question 1 in [11] and showed  $T^+$  maps  $L^1(w)$  into  $L^{1,\infty}(w)$  if  $w \in A_1^+$ .

The study of one-sided spaces emerged naturally alongside the study of one-sided operators. In one previous study, the authors studied one-sided BMO spaces associated with one-sided sharp functions and their relationship to good weights for the one-sided Hardy-Littlewood maximal functions [26]. It is well known that the classical Hardy spaces were the dual spaces of BMO spaces and were the natural alternative for Lebesgue spaces when  $p < 1$ . For some classical work on classical Hardy spaces, we refer to [6], [8], [9], [14] and [19]. In [36], Rosa and Segovia introduced the one-sided Hardy space  $H_+^q$  ( $0 < q \leq 1$ ). We first recall its definition. As usual,  $C_0^\infty(\mathbb{R})$  is the set of all functions with compact support having derivatives of all orders. For  $-\infty \leq r < \infty$ , we shall denote by  $\mathcal{S}(r, \infty)$  the space of all  $C_0^\infty(\mathbb{R})$  functions with support contained in  $(r, \infty)$  equipped with the usual topology and by  $\mathcal{S}'(r, \infty)$  the space of distributions on  $(r, \infty)$ . Given an integer  $\gamma \geq 1$  and  $x \in \mathbb{R}$ , we shall say that a  $C_0^\infty(\mathbb{R})$  function  $\psi(t)$  belongs to the class  $\Phi_\gamma(x)$  if there exists a bounded interval  $I_\psi = [x, \beta]$  containing the support of  $\psi(t)$  such that  $D^\gamma \psi(t)$  satisfies

$$|I_\psi|^{\gamma+1} \|D^\gamma \psi\|_{L^\infty} \leq 1.$$

Let  $f$  be a distribution in  $\mathcal{S}'(r, \infty)$ . One defines the one-sided maximal function  $M_{+,\gamma}^1(x)$  as

$$M_{+,\gamma}^1(x) = \sup\{|\langle f, \psi \rangle| : \psi \in \Phi_\gamma(x)\}$$

for every  $x > \gamma$ . We observe that  $M_{+,\gamma}^1(x)$  is a lower semicontinuous function.

Let  $w \in A_p^+$  and  $0 < q \leq 1$ . For every integral  $\gamma \geq 1$  satisfying  $(\gamma + 1)q \geq p > 1$  or  $(\gamma + 1)q > 1$  if  $p = 1$ , we shall say that the distribution  $f$  in  $\mathcal{S}'(x_{-\infty}, \infty)$  belongs to  $H_{+,\gamma}^q(w)$  if the “ $p$ -norm”

$$\|f\|_{H_{+,\gamma}^q(w)} = \left( \int_{x_{-\infty}}^\infty (M_{+,\gamma}^1(x))^q w(x) dx \right)^{1/q} < \infty.$$

It is easy to say that  $H_{+, \gamma}^q(w)$  is a Banach space [36].

If  $\psi \in \mathcal{S}$ ,  $\text{supp}(\psi) \subset (-\infty, 0]$ ,  $\int_{-\infty}^0 \psi(x)dx = 0$ ,  $f \in \mathcal{S}(x_{-\infty}, \infty)$  and  $x > x_{-\infty}$ , then one can define another one-sided maximal functions

$$M_{+, \psi}^2(x) = \sup_{t>0} |f * \psi_t(y)|,$$

where  $\psi_t(x) = t^{-1}\psi(x/t)$ . We shall say that  $f$  in  $\mathcal{S}'(x_{-\infty}, \infty)$  belongs to  $H_+^q(w)$  if

$$\|f\|_{H_+^q(w)} = \left( \int_{x_{-\infty}}^{\infty} (M_{+, \psi}^2(x))^q w(x) dx \right)^{1/q} < \infty,$$

again,  $0 < q \leq 1$  and  $w \in A_p^+(p \geq 1)$ . In [38, Theorem C], the authors proved that

$$H_+^q(w) = H_{+, \gamma}^q(w),$$

where  $w \in A_p^+(p \geq 1)$ ,  $0 < q \leq 1$  and  $q(\gamma + 1) > p$ .

There are still atomic decomposition for functions in  $H_+^q(w)$  ( $0 < q \leq 1$ ). We first recall the definition of  $H_+^q(w)$  atom [36]. A function  $a(x)$  defined on  $\mathbb{R}$  is called a  $q$ -atom with respect to  $w(x)$  if there exists an interval  $I$  (not necessary bounded) containing the support of  $a(x)$  such that

- (a)  $I \subset (x_{-\infty}, \infty)$  and  $w(I) < \infty$ ,
- (b)  $\|a\|_{L^\infty} \leq w(I)^{-1/q}$ ,
- (c)  $|I| < \text{dist}(x_{-\infty}, I)$ , and  $\int_I a(x)dx = 0$ .

We shall say that  $I$  is the interval associated to the atom  $a(x)$ .

Let  $w \in A_p^+(p \geq 1)$ ,  $\gamma \geq 1$  be an integer and  $0 < q \leq 1$  such that  $(\gamma + 1)q \geq p > 1$  or  $(\gamma + 1)q > 1$  if  $p = 1$ . Then, if  $f \in H_+^q(w)$ , there exists a sequence  $\{a_k(x)\}$  of  $q$ -atom with respect to  $w(x)$  and a sequence  $\{\lambda_k\}$  of real numbers such that

$$(1.2) \quad f = \sum \lambda_k a_k(x) \in \mathcal{S}'(x_{-\infty}, \infty)$$

and

$$\|f\|_{H_+^q(w)} \approx \sum |\lambda_k|^q.$$

The sum in (1.2) is both in the sense of distributions and in the  $H_+^q(w)$  norm [36, Theorem 2.2].

Besides one-sided maximal functions, Ombrosi and Segovia [29] studied the boundedness of the one-sided Calderon-Zygmund operator  $\tilde{T}^+$  on  $H_+^q(w)$  ( $0 < q \leq 1$ ) with  $w \in A_p^+(p \geq 1)$  under a generic condition and proved that  $\tilde{T}^+$  can be extended to bounded operators from  $H_+^q(w)$  to  $H_+^q(w)$ . In fact, they proved the boundedness in a more general case, see [29] for more details.

It is easy to check that  $H_+^1(w) \subset L^1(w)$ . Therefore, if  $T^+$  maps  $H_+^1(w)$  into  $L^1(w)$ , then we can prove the weighted  $L^p$  boundedness of  $T^+$  by a standard interpolation argument [30]. Therefore an interesting problem will be formulated as

**Question 2** Does  $T^+$  map  $H_+^1(w)$  into  $L^1(w)$  with  $w \in A_1^+$ ?

However, the above problem is still open even in the classical “two-sided” case (see [20]). In the present note, we partly answer this question when  $P(x, y) = P(x - y)$ . In

this case,  $T^+$  is a convolution operator, which can allow us to use Fourier transform. In fact, we can prove the following results.

**Theorem 1.1.** *Let  $P(x)$  be a polynomial which satisfies  $P'(0) = 0$  and  $w \in A_1^+$ . Then there exists a constant  $C > 0$ , which depends only on  $A_1^+(w)$  and the degree of  $P(x)$  (not its coefficients), such that*

$$\|T^+ f\|_{L^1(w)} \leq C \|f\|_{H_+^1(w)}$$

for all  $f \in H_+^1(w)$ .

Rosa and Segovia [35] also considered  $[H_+^q(w)]^*$ -the dual space of  $H_+^q(w)$  formed by all the real valued continuous linear functions  $F$  with the norm

$$\|F\| = \sup\{|F(f)| : \|f\|_{H_+^q(w)} \leq 1\}.$$

They gave a characterization of  $[H_+^q(w)]^*$  in terms of certain classes one-sided weighted *BMO* of Lipschitz spaces. We will touch only a few aspects of this theory and refer to [35] for more details. We can obtain

**Corollary 1.2.** *Let  $P$  and  $w$  be in Theorem 1.1. Then  $T^+$  is bounded from  $L^\infty(w)$  into  $[H_+^q(w)]^*$ .*

We would like to point out that the restriction  $P'(0) = 0$  in Theorem 1.1 is essential. For example, we take  $w = 1$ ,  $P(x) = \lambda x (\lambda > 0)$ ,  $f(x) = \pi^{-1} \lambda (\chi_{[0, \pi/2\lambda]}(x) - \chi_{[-\pi/2\lambda, 0]}(x))$  and  $K(x) = \frac{\sin(\log|x|)}{(x \log|x|)} \chi_{(-\infty, 0)}(x)$ . Let

$$T^+ f(x) = \text{p.v.} \int_x^\infty e^{iP(x-y)} K(x-y) f(y) dy.$$

Below we will let  $x < -100\lambda^{-1}$ . Let

$$g(x) = T^+ f(x) - e^{i\lambda x} K(x) \int_x^\infty e^{-i\lambda y} f(y) dy.$$

Then

$$|g(x)| \leq \int_x^\infty |K(x-y) - K(x)| |f(y)| dy \leq C\lambda/|x|^2,$$

which implies

$$\begin{aligned} |T^+ f(x)| &\geq \left| e^{i\lambda x} K(x) \int_x^\infty e^{-i\lambda y} f(y) dy \right| - |g(x)| \\ &\geq |K(x)| \left| 2\lambda/\pi \int_0^{\pi/2\lambda} \sin(\lambda y) dy \right| - C\lambda^{-1}|x|^{-2} \\ &= (2/\pi)|K(x)| - C\lambda^{-1}|x|^{-2}. \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} |T^+ f(x)| dx \geq (2/\pi) \int_{-\infty}^{-100\lambda^{-1}} |K(x)| dx - C/100 = \infty.$$

It is well known that for  $q < 1$ ,  $\tilde{T}^+$  is still bounded from  $H_+^q(w)$  to  $H_+^q(w)$  (see [29, Theorem 3.1]). However, this is no longer suitable for  $T^+$ . In Section 3 of this article, we will show that this fails even  $P(x, y)$  is the bilinear phase function by following a simple counterexample adopting from [31].

When  $1 < p < \infty$ , there is still an interesting question for general  $P(x, y)$ :

**Question 3** Is  $T^+$  bounded on  $L^p(w)$  ( $1 < p < \infty$ ) with  $w \in A_p^+$ ?

Recently, we proved the weighted  $L^p$  ( $1 < p < \infty$ ) estimates for  $T^+$  in [12] and showed that for any real polynomial  $P(x, y)$ ,  $T^+$  is of type  $(L^p(w), L^p(w))$  for  $w \in A_p^+$ . It is easy to see that when  $P(x, y)$  is trivial, for example,  $P(x, y) = 0$ , then  $T^+$  is  $\tilde{T}^+$ . In [2], the authors proved that  $\tilde{T}^+$  enjoys weighted  $L^p$  ( $1 < p < \infty$ ) boundedness properties similar to those of  $M^+$ . As a result of this close relationship between  $T^+$  and  $\tilde{T}^+$ , there is one meaningful problem: Are there some connections between the boundedness of these two one-sided operators? In this paper, we shall give a criterion for the weighted  $L^p$ -boundedness of  $T^+$  and show that the boundedness of  $T^+$  can be deduced by the corresponding boundedness of  $\tilde{T}^+$ .

**Theorem 1.3.** *Let  $P(x, y)$  be a real polynomial,  $K$  be a one-sided Calderón-Zygmund kernel and  $b(r)$  be a bounded variation function on  $[0, \infty)$ . For  $1 < p < \infty$  and  $w \in A_p^+$ , we have*

(a) *The operator*

$$\tilde{T}^{+,b}f(x) = p.v \int_x^\infty b(y-x)K(x-y)f(y)dy$$

*is of type  $(L^p(w), L^p(w))$ .*

(b) *The operator*

$$T^{+,b}f(x) = p.v \int_x^\infty e^{iP(x,y)}b(y-x)K(x-y)f(y)dy$$

*is of type  $(L^p(w), L^p(w))$ . Here its norm depend only on the total degree of  $P(x, y)$  and  $A_p^+(w)$ , but not on the coefficients of  $P(x, y)$ .*

Furthermore, we have

**Theorem 1.4.** *Let  $w$ ,  $p$  and  $K$  be as in Theorem 1.3. Then the following statements are equivalent:*

(a) *If  $P(x, y)$  is a nontrivial polynomial ( $P(x, y)$  does not take the form  $P_0(x) + P_1(y)$ , where  $P_0$  and  $P_1$  are polynomials defined on  $\mathbb{R}$ ), then the operator  $T^+$  is of type  $(L^p(w), L^p(w))$ .*

(b) *If  $P(x, y)$  satisfies  $P(x, y) = P(x-h, y-h) + P_0(x, h) + P_1(y, h)$  with  $h \in \mathbb{R}$  and  $P_0$  and  $P_1$  are polynomials defined on  $\mathbb{R}$ , then the operator  $T^+$  is of type  $(L^p(w), L^p(w))$ .*

(c) *The truncated operator*

$$\tilde{T}_0^+f(x) = p.v \int_x^{x+1} K(x-y)f(y)dy$$

*is of type  $(L^p(w), L^p(w))$ .*



Theorem 1.3 and Theorem 1.4 readily produces the following result for the maximal operator corresponding to  $T^+$ :

**Theorem 1.5.** *Let  $w$ ,  $p$  and  $K$  be as in Theorem 1.3. Then the maximal operator*

$$T_*^+ f(x) = \sup_{\varepsilon > 0} \left| \int_{x+\varepsilon}^{\infty} e^{iP(x,y)} K(x-y) f(y) dy \right|$$

*is of type  $(L^p(w), L^p(w))$ , where its norm depends only on the total degree of  $P(x, y)$ , but not on the coefficients of  $P(x, y)$ .*

We end this section with the outline of this paper. Section 2 contains the proof of Theorem 1.1 and a counterexample to show that the boundedness for  $T^+$  in Theorem 1.1 can not be extended to  $H_+^q(w)$  when  $q < 1$  and  $w \in A_p^+(1 < p < \infty)$ . In Section 3, the proofs of Theorem 1.3-Theorem 1.5 will be given.

## 2. ONE-SIDED ESTIMATES ON WEIGHTED HARDY SPACES

In order to prove Theorem 1.1, we first collect some lemmas. If  $w(x) \in A_p$ , then it is a doubling weight, that is, there exists  $C > 0$  such that

$$\int_{a-2h}^{a+2h} w \leq C \int_{a-h}^{a+h} w$$

for all  $a \in \mathbb{R}$  and  $h > 0$ . However, one-sided weights  $A_p^+$  do not satisfy this property. But the weights  $A_p^+$  satisfy a one-sided doubling condition:

**Lemma 2.1.** [34] *Let  $w(x) \in A_p^+(p \geq 1)$ . Then there exists a constant  $C > 0$  such that*

$$\int_{a-h}^{a+h} w \leq C \int_a^{a+h} w$$

for all  $a \in \mathbb{R}$  and  $h > 0$ .

For  $\lambda > 1$ , we denote by  $I^- = [a - h, a]$  and  $\lambda I^- = [a - \lambda h, a]$ . Therefore, for  $w \in A_1^+$ , we have

$$(2.1) \quad w(\lambda I^-) \leq C \lambda w(I^-)$$

by Lemma 2.1, see also [41, Proposition 12].

Besides the doubling condition,  $A_p$  weights satisfy the reverse Hölder inequality which play a key role in the proof of the strong type  $(p, p)$  inequality of operators. If  $w \in A_p$ , then there exists  $\delta > 0$  and  $C > 0$  such that

$$\frac{1}{b-a} \int_a^b w^{1+\delta} \leq \left( \frac{1}{b-a} \int_a^b w \right)^{1+\delta}$$

for all intervals  $(a, b)$ . Unfortunately, one-sided weights  $A_p^+$  do not satisfy the reverse Hölder inequality. However, a substitute was found in [24]:



**Lemma 2.2.** *If  $w \in A_p^+(p \geq 1)$ , then there exists constants  $C$  and  $\delta$  such that for all  $a$  and  $b$*

$$(2.2) \quad \int_a^b w^{1+\delta} \leq C (M^-(w\chi_{(a,b)})(b))^\delta \int_a^b w.$$

(2.2) implies that

$$M^-(w^{1+\delta}\chi_{(a,b)})(b) \leq C (M^-(w\chi_{(a,b)})(b))^{1+\delta},$$

which is what we have called the weak reverse Hölder inequality since

$$(M^-(w\chi_{(a,b)})(b))^{1+\delta} \leq M^-(w^{1+\delta}\chi_{(a,b)})(b)$$

by the Hölder inequality.

We point out that it was proved in [24] that (2.2) holds if and only if there exists positive numbers  $\delta$  and  $C$  such that

$$(2.3) \quad \frac{1}{c-a} \int_a^c w^{1+\delta} \leq C \left( \frac{1}{b-a} \int_a^b w \right)^{1+\delta}$$

for all numbers  $a < b$  and  $c = (a+b)/2$ , which seems to be a more natural formulation.

Let  $w \in A_1^+$ ,  $I^- = [x_0 - h, x_0]$ , and  $a(x)$  be a  $H_+^1(w)$  atom, which satisfies

- (a)  $\text{supp}(a) \subset I^-$ ;
- (b)  $\int_{I^-} a(x) dx = 0$ ;
- (c)  $\|a\|_{L^\infty} \leq w(I^-)^{-1}$ .

Let  $I_0^- = [-1, 0]$ . Then we have  $w_0(x) = w(x_0 + hx)$ . It is easy to see that  $w_0 \in A_1^+$  and  $A_p^+(w_0) = A_p^+(w)$ . Set  $b(x) = ha(x_0 + hx)$ , we see that  $b(x)$  is a  $H_+^1(w_0)$  atom, and it satisfies

$$(2.4) \quad \text{supp}(b) \subset I_0^-,$$

$$(2.5) \quad \int_{I_0^-} b(x) dx = 0,$$

$$(2.6) \quad \|b\|_{L^\infty} \leq w_0(I_0^-)^{-1}.$$

Furthermore, we have

$$(T^+a)(x_0 + hx) = h^{-1}(T_1^+b)(x),$$

where  $T_1^+f(x) = \int_{I_0^-} e^{iP(hx-hy)} K(x-y)b(y)dy$ , which leads to  $\|T^+a\|_{L^1(w)} = \|T_1^+b\|_{L^1(w)}$ .

To prove Theorem 1.1, we first prove the following proposition.

**Proposition 2.3.** *Let  $P(x)$  be a polynomial with  $P'(0) = 0$  and  $w \in A_1^+$ . Then for any  $H_1^+(w)$  atom  $a(x)$ , we have*

$$\|T^+(a)\|_{L^1(w)} \leq C,$$

where  $C$  is a constant, depending only on the degree of  $P(x)$  and  $A_1^+(w)$ .

The preceding argument shows that it is sufficient to prove Proposition 2.3 for  $H_1^+(w)$  atoms which satisfy (2.4)-(2.6) (with  $w(x)$  replaced by  $w_0(x)$ ). We first list a few lemmas that are needed in the proof of Proposition 2.3.

**Lemma 2.4.** [33] *Let  $Q(x) = \sum_{\alpha \leq d} q_\alpha x^\alpha$  be a polynomial in  $x \in \mathbb{R}$ , with degree  $d$ . Suppose  $\varepsilon < 1/d$ . Then*

$$\int_{|x| \leq 1} |Q(x)|^{-\varepsilon} dx \leq A_\varepsilon \left( \sum_{\alpha \leq d} |q_\alpha| \right)^{-\varepsilon}.$$

**Lemma 2.5.** [33] *Let  $\psi \in C^1[\alpha, \beta]$ ,  $\varepsilon = \min\{1/a_1, 1/n\}$ ,  $\lambda > 0$ . Then*

$$\left| \int_\alpha^\beta e^{i\lambda\phi(t)} \psi(t) dt \right| \leq C\lambda^{-\varepsilon} \left\{ \sup_{\alpha \leq t \leq \beta} |\psi(t)| + \int_\alpha^\beta |\psi'(t)| dt \right\},$$

where  $\phi$  is real-valued phase of the form  $\phi(t) = t^{a_1} + \mu_2 t^{a_2} + \cdots + \mu_n t^{a_n}$  with real parameters  $\mu_2, \dots, \mu_n$  and distinct positive exponents  $a_1, a_2, \dots, a_n$ .

**Lemma 2.6.** *Let  $P(x)$  be a polynomial of degree  $m(m \geq 2)$  and  $P(x) = \sum_{\alpha \leq m} a_\alpha x^\alpha$ . Suppose  $\varphi$  and  $\psi$  are two functions in  $C_0^\infty$ . Define  $T_j^+$  by*

$$(T_j^+ f)(x) = \psi(2^{-j}x) \int_x^\infty e^{iP(x-y)} \varphi(y) f(y) dy.$$

Then we have

$$\|T_j^+ f\|_{L^2(\mathbb{R})} \leq C |a_m|^{-1/(4(m-1))} 2^{j/4} \|f\|_{L^2(\mathbb{R})}.$$

Combining Lemma 2.4 with Lemma 2.5, we can prove Lemma 2.6 by a similar analysis as in [15], corresponding argument, see also [11] and [33].

The following proposition about one-sided Calderó-Zygmund  $\tilde{T}^+$  play a key role in the proof of Proposition 2.3.

**Proposition 2.7.** *Let  $\tilde{T}^+$  be a one-sided Calderó-Zygmund operator and  $a(x)$  be a  $H_+^1(w)$  atom satisfy (2.4)-(2.6). Then*

$$\|\tilde{T}^+ a\|_{L^1(w)} \leq C.$$

*Proof.* Let  $\text{supp } a \subset I_0^- = [-1, 0]$  and  $\tilde{I}^- = 2I_0^- = [-2, 0]$ . Then for  $x \in \tilde{I}^{-,c} \equiv (x_{-\infty}, \infty) \setminus \tilde{I}^-$ , we have

$$\begin{aligned} |\tilde{T}^+ a(x)| &\leq \int_x^\infty |K(x-y) - K(x)| |a(y)| dy \\ &\leq \frac{1}{|x|^2} \int_{I_0^-} |y| |a(y)| dy \\ &\leq \frac{1}{|x|^2} w(I_0^-)^{-1}. \end{aligned}$$

This implies

$$\begin{aligned}
\int_{\tilde{I}^-,c} |\tilde{T}^+ a(x)| w(x) dx &\leq C \int_{\tilde{I}^-,c} \frac{1}{|x|^2} w(I_0^-)^{-1} w(x) dx \\
&\leq C w(I_0^-)^{-1} \int_{x=-\infty}^{-2} \frac{w(x)}{|x|^2} dx \\
&\leq C w(I_0^-)^{-1} \sum_{j=1}^{\infty} 2^{-2j} \int_{-2^{j+1}}^0 w(x) dx \\
&\leq C \sum_{j=1}^{\infty} 2^{-j} \leq C,
\end{aligned}$$

where we use (2.1) and (2.6).

For  $x \in \tilde{I}^-$ , it is easy to get

$$\begin{aligned}
\int_{\tilde{I}^-} |\tilde{T}^+ a(x)| w(x) dx &\leq \left( \int_{\mathbb{R}} |T^+ a|^2 w(x) dx \right)^{1/2} \left( \int_{-2}^0 w(x) dx \right)^{1/2} \\
&\leq C \left( \int_{\mathbb{R}} |a|^2 w(x) dx \right)^{1/2} w(\tilde{I}^-)^{1/2} \\
&\leq C \|a\|_{L^\infty} w(I_0^-) \leq C.
\end{aligned}$$

□

We now come back to the proof of Proposition 2.3. Assume that  $a$  is a  $H_+^1(w)$  atom that satisfies (2.4)-(2.6). Adopting the idea in [15], we shall prove Proposition 2.3 by using induction on  $m$ , the degree of  $P(x)$ . When  $m = 0$ , that is  $P(x) = 0$ , which imply  $T^+ = \tilde{T}^+$  in that case. Proposition 2.3 holds by Proposition 2.7. We now assume that Proposition 2.3 is true for  $\deg(P) \leq m - 1$ . The task is now to show Proposition 2.3 for  $\deg(P) = m$ . We write

$$P(x - y) = a_m(x - y)^m + P_{m-1}(x - y),$$

where  $\deg(P_{m-1}) \leq m - 1$ . Let  $b = \max\{|a_m|^{-1/(m-1)}, 2\}$ . We distinguish two cases to obtain our desired results.

*Case 1.*  $b < -x_{-\infty}$ . In this case, we break the integral into three parts:

$$\begin{aligned}
\|T^+ a\|_{L^1(w)} &\leq \left| \int_{|x| \leq b} T^+ a(x) w(x) dx \right| + \left| \int_{x=-\infty}^{-b} T^+ a(x) w(x) dx \right| + \left| \int_b^\infty T^+ a(x) w(x) dx \right| \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

The first step is to show that  $I_1 \leq C$ . If  $b = 2$ , the estimates follows from a standard argument as

$$\begin{aligned}
I_1 &= \left| \int_{|x| \leq 2} T^+ a(x) |w(x) dx \right| \\
&\leq \|T^+(a)\|_{L^2(w)} \left( \int_{-2}^0 w(x) dx \right)^{1/2} \\
&\leq C \|a\|_{L^2(w)} w(I_0^-)^{1/2} \\
&\leq C \|a\|_{L^\infty} w(I_0^-) \\
&\leq C,
\end{aligned}$$

where we use (2.1), (2.6) and the weighted  $L^p$  estimate for  $T^+$  ([12]).

Assuming  $b = |a_m|^{-1/(m-1)}$ , by the above argument for  $b = 2$ , we have

$$\begin{aligned}
I_1 &\leq \left| \int_{|x| \leq 2} T^+ a(x) |w(x) dx \right| + \left| \int_{2 \leq |x| \leq b} T^+ a(x) |w(x) dx \right| \\
&\leq C + \left| \int_{-b}^{-2} \left| \int_{x-\infty}^{\infty} e^{iP_{m-1}(x-y)} K(x-y) a(y) dy \right| w(x) dx \right| \\
&\quad + \left| \int_{-b}^{-2} \left| \int_{x-\infty}^{\infty} e^{i(P(x-y) - P_{m-1}(x-y) - a_m x^m)} K(x-y) a(y) dy \right| w(x) dx \right| \\
&=: C + J_1 + J_2.
\end{aligned}$$

By inductive hypothesis,  $J_1 \leq C$ . On the other hand, we have

$$\begin{aligned}
J_2 &\leq C |a_m| \int_{-b}^{-2} \int_{I_0^-} \frac{|(x-y)^m - x^m|}{y-x} |a(y)| dy w(x) dx \\
&\leq C |a_m| \int_{-b}^{-2} |x|^{m-2} w(x) dx \int_{I_0^-} |a(y)| dy \\
&\leq C |a_m| \sum_{j \geq 1, 2^j \leq b} 2^{j(m-2)} \int_{-b}^0 w(x) dx \int_{I_0^-} |a(y)| dy \\
&\leq C |a_m| b^{m-1} \\
&\leq C.
\end{aligned}$$

Next, we prove that  $I_2 \leq C$ . Assume that  $2^{j_0} \leq b \leq 2^{j_0+1}$ . Let  $\varphi \in C_0^\infty(\mathbb{R})$  and  $\varphi = 1$  on  $I_0^-$ . Choosing  $\psi \in C_0^\infty(\mathbb{R})$  such that  $\text{supp}(\psi) \subset \{1/4 < |x| < 4\}$ ,  $\psi \geq 0$  and  $\psi(x) = 1$ , for  $1 \leq |x| \leq 2$ . We have

$$\begin{aligned}
I_2 &\leq \int_{x-\infty}^{-b} \int_x^\infty |K(x-y) - K(x)| |a(y)| dy w(x) dx + \int_{x-\infty}^{-b} \left| \int_x^\infty e^{iP(x-y)} a(y) dy K(x) w(x) \right| dx \\
&=: K_1 + K_2.
\end{aligned}$$

Since  $K$  is one-sided Calderó-Zygmund kernel, by (2.6), we can estimate  $K_1$  as

$$\begin{aligned} K_1 &\leq \int_{x-\infty}^{-2} \|a\|_{L^\infty} \int_{-1}^0 dy \frac{w(x)}{|x|^2} dx \\ &\leq \frac{C}{w(I_0^-)} \int_{x-\infty}^{-2} \frac{w(x)}{|x|^2} dx \\ &\leq \frac{C}{w(I_0^-)} \sum_{j=1}^{\infty} 2^{-2j} w(2^{j+1} I_0^-) \\ &\leq C. \end{aligned}$$

While the Hölder inequality allows us to estimate  $K_2$  as

$$\begin{aligned} K_2 &\leq \sum_{j \geq j_0} \int_{-2^{j+1}}^{-2^j} \frac{w(x)}{|x|} \psi(2^{-j}x) \left| \int_x^\infty e^{iP(x-y)} \varphi(y) a(y) dy \right| dx \\ &= \sum_{j \geq j_0} \int_{-2^{j+1}}^{-2^j} \frac{w(x)}{|x|} |T_j^+(a)| dx \\ &\leq \sum_{j \geq j_0} \|T_j^+(a)\|_{L^p} \left( \int_{-2^{j+1}}^{-2^j} \frac{w(x)^{1+\varepsilon}}{|x|^{1+\varepsilon}} dx \right)^{1/(1+\varepsilon)}. \end{aligned}$$

Invoking the properties (a) of  $A_1^+$  weights and (2.3), we obtain

$$\int_{-2^{j+1}}^{-2^j} \frac{w(x)^{1+\varepsilon}}{|x|^{1+\varepsilon}} dx \leq C 2^{-j\varepsilon} \frac{2^{j+1} w^{1+\varepsilon}(I_0^-)}{|2^{j+1} I_0^-|} \leq C 2^{-j\varepsilon} w^{1+\varepsilon}(I_0^-).$$

Therefore,

$$I_2 \leq C + \sum_{j \geq j_0} \|T_j^+(a)\|_{L^p} 2^{-j\varepsilon/(1+\varepsilon)} w(I_0^-).$$

After noting that

$$\|T_j^+(a)\|_{L^\infty} \leq C \|a\|_{L^\infty},$$

by Lemma 2.6 and interpolation we get

$$\begin{aligned} \|T_j^+(a)\|_{L^p} &\leq 2^{j/p} (|a_m| 2^{j(m-1)})^{-1/(2p(m-1))} \left( \int_{I_0^-} |a|^p \right)^{1/p} \\ &\leq 2^{j/2p} (|a_m|)^{-1/(2p(m-1))} \|a\|_{L^\infty} \\ &\leq 2^{j/2p} (|a_m|)^{-1/(2p(m-1))} w(I_0^-)^{-1}. \end{aligned}$$

The estimate for  $I_2$  is completed by showing that

$$\begin{aligned}
I_2 &\leq C + \sum_{j \geq j_0} 2^{j/2p} |a_m|^{-1/2p(m-1)} w(I_0^-)^{-1} 2^{-j\varepsilon/(1+\varepsilon)} w(I_0^-) \\
&\leq C + |a_m|^{-1/2p(m-1)} 2^{-j_0/2p} \\
&\leq C + |a_m|^{-1/2p(m-1)} b^{-1/2p} \\
&\leq C.
\end{aligned}$$

On the other hand,  $\text{supp } K = (-\infty, 0)$  and  $\text{supp } a \subset I_0^-$  show that

$$I_3 = 0.$$

We conclude from above estimate for  $I_1$ ,  $I_2$  and  $I_3$  that

$$\|T^+(a)\|_{L^1(w)} \leq C.$$

*Case 2.*  $b > -x_{-\infty}$ . In this case, we have

$$\begin{aligned}
\|T^+(a)\|_{L^1(w)} &\leq \left| \int_{x_{-\infty}}^b T^+(a)(x) w(x) dx \right| + \left| \int_b^\infty T^+(a)(x) w(x) dx \right| \\
&=: \tilde{I}_1 + \tilde{I}_2.
\end{aligned}$$

Similar as in the estimate of  $I_3$ , we have  $\tilde{I}_2 = 0$ . So, we only need to consider  $\tilde{I}_1$ . If  $b = 2$ , by Lemma 2.1 and (2.6), we have

$$\begin{aligned}
\tilde{I}_1 &\leq \|T^+(a)\|_{L^2(w)} \left( \int_{x_{-\infty}}^0 w(x) dx \right)^{1/2} \\
&\leq \|a\|_{L^2(w)} \left( \int_{-1}^0 w(x) dx \right)^{1/2} \\
&\leq \|a\|_{L^\infty} w(I_0^-) \\
&\leq C.
\end{aligned}$$

If  $b = |a_m|^{-1/(m-1)}$ . Applying the above estimates for  $\tilde{I}_1$  when  $b = 2$ , we have

$$\begin{aligned}
\tilde{I}_1 &\leq \left| \int_{x_{-\infty}}^2 T^+(a) w(x) dx \right| + \left| \int_2^b T^+(a) w(x) dx \right| \\
&\leq C + \tilde{J}_1 \\
&\leq C
\end{aligned}$$

as a result of  $\tilde{J}_1 = 0$ .

Combining *Case 1* and *Case 2*, we have thus proved Proposition 2.3.

Having disposed of the above preliminaries, we can now return to the proof of our main theorem. As a byproduct of Proposition 2.3, we have

$$\|T^+ f\|_{L^1(w)} \leq \sum_{j=1}^{\infty} |\lambda_j| \|T^+(a_j)\|_{L^1(w)} \leq C \sum_{j=1}^{\infty} |\lambda_j| \leq C \|f\|_{H_+^1(w)}.$$

We have completed the proof of Theorem 1.1.

Inspired by the main idea from [31], in this section, a counterexample is given to show that the  $H_+^1(w)$  theory on the one-sided oscillatory singular operators can not be extended to the  $H_+^q(w)$  case, if  $q < 1$ . Let  $\overline{T}^+$  be defined as

$$\overline{T}^+ f(x) = p.v. \int_{x-\infty}^{\infty} e^{ixy} \frac{f(y)}{x-y} dy.$$

Take  $\delta > 0$  very small, and  $\text{supp } f \subset I_\delta = [-\delta, \delta]$  given by

$$f(y) = \begin{cases} (2\delta)^{-1/q}, & y \in [\delta/2, \delta]; \\ (-2\delta)^{-1/q}, & y \in [-\delta, -\delta/2]; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that  $|f| \leq |I_\delta|^{-1/q}$ ,  $\int_{I_\delta} f(y) dy = 0$ . Therefore, we have

$$\text{Im}(T^+ a)(x) = (2\delta)^{-1/q} \left( \int_{\delta/2}^{\delta} \frac{\sin(xy)}{x-y} dy + \int_{\delta/2}^{\delta} \frac{\sin(xy)}{x+y} dy \right).$$

Let  $x \in (-\pi/3\delta, -\pi/4\delta)$ . Then  $x-y < 0$ ,  $x+y < 0$  for any  $y \in [\delta/2, \delta]$ . Also, we have  $-\pi/3 < xy < -\pi/8$ . Thus

$$\begin{aligned} |\text{Im}(T^+ a)(x)| &> C(2\delta)^{-1/q} \left( \int_{\delta/2}^{\delta} \frac{dy}{y-x} + \int_{\delta/2}^{\delta} \frac{dy}{x-y} \right) \\ &= C\delta^{-1/q} \ln\left(1 + \frac{-\delta x}{x^2 + \delta/2x - \delta^2/2}\right) \geq C\delta^{-1/q}(-x)^{-1}. \end{aligned}$$

Therefore, we have

$$\int_{\mathbb{R}} |T^+ a(x)|^q dx \geq \int_{-\pi/3\delta}^{-\pi/4\delta} \delta^{q-1} (-x)^{-q} dx = \delta^{2(q-1)}.$$

Set  $\max\{-1, 2(q-1)\} < \alpha \leq 0$ . Then  $w = |x|^\alpha \in A_1^+$  and

$$\int_{\mathbb{R}} |T^+ a(x)|^q w(x) dx \geq C\delta^{q-1} \int_{\pi/4\delta}^{\pi/3\delta} t^{\alpha-q} dt \geq C\delta^{2(q-1)-\alpha} \rightarrow \infty$$

by letting  $\delta \rightarrow 0$  since  $q < 1$ .

### 3. CRITERION ON WEIGHTED $L^p$ ESTIMATES

In this section, a criterion on boundedness of the one-sided operators mentioned in Section 1 and its effects on weighted  $L^p$  spaces are described. Let us first begins with some properties about the  $A_p^+$  classes, which will be used in the proof of our main results.

**Lemma 3.1.** [12] *Let  $1 < p < \infty$  and  $w \in A_p^+$ . Then*

- (a)  $A_p^+(\delta^\lambda(w)) = A_p^+(w)$ , where  $\delta^\lambda(w)(x) = w(\lambda x)$  for all  $\lambda > 0$ .
- (b) there exists  $\varepsilon > 0$  such that  $w^{1+\varepsilon} \in A_p^+$ .

The following celebrated interpolation theorem of operators with change of measures will be needed in our analysis.



**Lemma 3.2.** [43] *Suppose that  $u_0, v_0, u_1, v_1$  are positive weight functions and  $1 < p_0, p_1 < \infty$ . Assume sublinear operator  $S$  satisfies:*

$$\|Sf\|_{L^{p_0}(u_0)} \leq C_0 \|f\|_{L^{p_0}(v_0)},$$

and

$$\|Sf\|_{L^{p_1}(u_1)} \leq C_1 \|f\|_{L^{p_1}(v_1)}.$$

Then

$$\|Sf\|_{L^p(u)} \leq C \|f\|_{L^p(v)}$$

holds for any  $0 < \theta < 1$  and  $1/p = \theta/p_0 + (1 - \theta)/p_1$ , where  $u = u_0^{(p\theta)/p_0} u_1^{p(1-\theta)/p_1}$ ,  $v = v_0^{(p\theta)/p_0} v_1^{p(1-\theta)/p_1}$  and  $C \leq C_0^\theta C_1^{1-\theta}$ .

To prove Theorem 1.3, the following lemma is still needed:

**Lemma 3.3.** *Suppose that  $1 < p < \infty$ ,  $w \in A_p^+$  and  $K$  satisfies  $|K(x, y)| \leq C/(y - x)$ . If the operator*

$$\overline{T}^+ = p.v \int_x^\infty K(x, y) f(y) dy$$

is of type  $(L^p(w), L^p(w))$ , then the operator

$$\overline{T}_\varepsilon^+ f(x) = \int_x^{x+\varepsilon} K(x, y) f(y) dy$$

is of type  $(L^p(w), L^p(w))$ .

*Proof.* For  $h \in \mathbb{R}$ , decompose  $f$  into three parts as

$$\begin{aligned} f(y) &= f\chi_{\{|y-h|<\varepsilon/2\}}(y) + f\chi_{\{\varepsilon/2 \leq |y-h| < 5\varepsilon/4\}}(y) + f\chi_{\{|y-h| \geq 5\varepsilon/4\}}(y) \\ &=: f_1(y) + f_2(y) + f_3(y). \end{aligned}$$

When  $|x - h| < \varepsilon/4$ , it is easy to show  $\overline{T}_\varepsilon^+ f_1(x) = \overline{T}^+ f_1(x)$ , which allows the following to be true

$$(3.1) \quad \int_{|x-h|<\varepsilon/4} |\overline{T}_\varepsilon^+ f_1(x)|^p w(x) dx \leq \int_{\mathbb{R}} |\overline{T}^+ f_1(x)|^p w(x) dx \leq C \int_{|y-h|<\varepsilon/2} |f(y)|^p w(y) dy,$$

where  $C$  is independent of  $h$  and the coefficients of  $P(x, y)$ .

The fact that if  $|x - h| < \varepsilon/4$  and  $\varepsilon/2 \leq |y - h| < 5\varepsilon/4$ , then  $\varepsilon/4 < y - x < 3\varepsilon/2$  allows the following to be shown

$$|\overline{T}_\varepsilon^+ f_2(x)| \leq C \int_{x+\varepsilon/4}^{x+\varepsilon} \frac{1}{(y-x)} |f_2(y)| dy \leq CM^+(f_2)(x).$$

On account of the boundedness of  $M^+$ , the following can be proved

$$(3.2) \quad \int_{|x-h|<\varepsilon/4} |\overline{T}_\varepsilon^+ f_2(x)|^p w(x) dx \leq C \int_{|y-h|<5\varepsilon/4} |f(y)|^p w(y) dy,$$

where  $C$  is independent of  $h$  and the coefficients of  $P(x, y)$ .

Again notice that if  $|x - h| < \varepsilon/4$  and  $|y - h| \geq 5\varepsilon/4$ , then  $y - x > \varepsilon$ , the following can be shown

$$(3.3) \quad \overline{T}_\varepsilon^+ f_3(x) = 0.$$

Combining (3.1), (3.2) and (3.3), the following

$$\int_{|x-h|<\varepsilon/4} |\overline{T}_\varepsilon^+ f(x)|^p w(x) dx \leq C \int_{|y-h|<5\varepsilon/4} |f(y)|^p w(y) dy.$$

holds uniformly in  $h \in \mathbb{R}$ , where  $C$  is independent of  $h$  and the coefficients of  $P(x, y)$ , which implies

$$\|\overline{T}_\varepsilon^+ f\|_{L^p(w)} \leq C \|f\|_{L^p(w)},$$

where  $C$  is independent of the coefficients of  $P(x, y)$ .  $\square$

Having disposed of the above preliminary steps, the proofs of Theorem 1.3-Theorem 1.5 can be addressed.

**3.1. Proof of Theorem 1.3.** Using the same method in [2], the proof of (a) can be easily obtained. We omit it's proof here.

The proof of (b) is similar to that of Theorem 1.5 in [12]. This argument can now be applied again for the completeness of this paper. Suppose  $P(x, y)$  is a real polynomial with degree  $k$  in  $x$  and degree  $l$  in  $y$ . We shall carry out the argument by induction. For any nonzero real polynomial  $P(x, y)$  in  $x$  and  $y$ , there are  $k, l, m \geq 0$  such that

$$(3.4) \quad P(x, y) = a_{kl} x^k y^l + R(x, y)$$

with  $a_{kl} \neq 0$  and

$$R(x, y) = \sum_{0 \leq \alpha < k, 0 \leq \beta \leq m} a_{\alpha\beta} x^\alpha y^\beta + \sum_{0 \leq \beta < l} a_{k\beta} x^k y^\beta.$$

We shall write  $d_x(P) = k$  and  $d_y(P) = l$ . Below we shall carry out the argument by using a double induction on  $k$  and  $l$ .

If  $d_x(P) = 0$  and  $d_y(P)$  is arbitrary, then  $P(x, y) = P(y)$  and  $T^+ f$  can be written as

$$T^+ f(x) = \lim_{\varepsilon \rightarrow 0^+} \int_{x+\varepsilon}^{\infty} K(x-y) g(y) dy$$

where  $g(y) = e^{iP(y)} f(y)$ . Therefore, the conclusion of Theorem 1.3 follows from the assumption.

Let  $k \geq 1$  and assume that the conclusion of Theorem 1.3 holds for all  $P(x, y)$  with  $d_x(P) \leq k-1$  and  $d_y(P)$  arbitrary.

We will now prove that the conclusion of Theorem 1.3 holds for all  $P(x, y)$  with  $d_x(P) = k$  and  $d_y(P)$  arbitrary.

If  $d_x(P) = k$  and  $d_y(P) = 0$ , then

$$P(x, y) = a_{k0} x^k + Q(x, y)$$

with  $d_x(Q) \leq k-1$ . By taking the factor  $e^{ia_{k0}x^k}$  out of the integral sign, we see that this case follows from the above inductive hypothesis.

Suppose  $l \geq 1$  and the desired bound holds when  $d_x(P) = k$  and  $d_y(P) \leq l - 1$ . Now, let  $P(x, y)$  be a polynomail with  $d_x(P) = k$  and  $d_y(P) = l$ , as given in (3.4).

*Case 1.*  $|a_{kl}| = 1$ .

Write

$$\begin{aligned} T^{+,b}f(x) &= \int_x^{1+x} e^{iP(x,y)} b(y-x) K(x-y) f(y) dy \\ &\quad + \sum_{j=1}^{\infty} \int_{2^{j-1}+x}^{2^j+x} e^{iP(x,y)} b(y-x) K(x-y) f(y) dy \\ &=: T_0^{+,b}f(x) + T_{\infty}^{+,b}f(x) \\ &=: T_0^{+,b}f(x) + \sum_{j=1}^{\infty} T_j^{+,b}f(x). \end{aligned}$$

Take any  $h \in \mathbb{R}$ , and write

$$P(x, y) = a_{kl}(x-h)^k(y-h)^l + R(x, y, h),$$

where the polynomial  $R(x, y, h)$  satisfies the induction assumption, and the coefficients of  $R(x, y, h)$  depend on  $h$ .

The estimates for  $T_0^{+,b}$  is given first. It is easy to confirm that

$$\begin{aligned} T_0^{+,b}f(x) &= \int_x^{1+x} e^{i(R(x,y,h)+a_{kl}(y-h)^{k+l})} b(y-x) K(x-y) f(y) dy \\ &\quad + \int_x^{1+x} \left\{ e^{iP(x,y)} - e^{i(R(x,y,h)+a_{kl}(y-h)^{k+l})} \right\} b(y-x) K(x-y) f(y) dy \\ &=: T_{01}^{+,b}f(x) + T_{02}^{+,b}f(x). \end{aligned}$$

Note that  $\|b\|_{\infty} < +\infty$ , by the induction assumption and Lemma 3.3,  $T_{01}^{+,b}$  is a  $(L^p(w), L^p(w))$  type operator and the norm depends on  $\|b\|_{\infty}$ , but not on the coefficients of  $P(x, y)$  and  $h$ .

The estimate for the term  $T_{02}^{+,b}$  can now be introduced. Evidently, if  $|x-h| < 1/4$  and  $0 < y-x < 1$ , then

$$|e^{iP(x,y)} - e^{i(R(x,y,h)+a_{kl}(y-h)^{k+l})}| \leq |a_{kl}| |x-y| = C(y-x).$$

Therefore, when  $|x-h| < 1/4$ , the following is true:

$$|T_{02}^{+,b}f(x)| \leq C\|b\|_{\infty} \int_x^{x+1} |f(y)| dx \leq CM^+(f(\cdot)\chi_{B(h, \frac{5}{4})}(\cdot))(x).$$

It follows that

$$\int_{|x-h| < 1/4} |T_{02}^{+,b}f(x)|^p w(x) dx \leq C\|b\|_{\infty}^p \int_{|y-h| < 5/4} |f(y)|^p w(y) dy$$

holds uniformly in  $h \in \mathbb{R}$ , which implies

$$(3.5) \quad \|T_0^{+,b}f\|_{L^p(w)} \leq C\|f\|_{L^p(w)},$$

where  $C$  is independent of the coefficients of  $P(x, y)$ .

The estimates for  $T_\infty^{+,b}f$  can now be given. For  $j \geq 1$ , the following can be shown

$$|T_j^{+,b}f(x)| \leq \int_{2^{j-1}+x}^{2^j+x} \frac{|f(y)|}{|x-y|} b(y-x) dy \leq C \|b\|_\infty M^+(f)(x),$$

where  $C$  is independent of  $j$ . By lemma 3.1, there exists  $\varepsilon > 0$ , such that  $w^{1+\varepsilon} \in A_p^+$ . Thus

$$(3.6) \quad \|T_j^{+,b}f\|_{L^p(w^{1+\varepsilon})} \leq C \|f\|_{L^p(w^{1+\varepsilon})},$$

where  $C$  is independent of  $j$ . On the other hand, recall Lemma 3.7 in [11] to see that

$$(3.7) \quad \|T_j^{+,b}f\|_{L^p} \leq C 2^{-j\delta} \|f\|_{L^p},$$

where  $C$  is dependents only on the total degree of  $P(x, y)$  and  $\delta > 0$ . From (3.6), (3.7) and Lemma 3.2, it follows that

$$(3.8) \quad \|T_j^{+,b}f\|_{L^p(w)} \leq C 2^{-j\theta\delta} \|f\|_{L^p(w)},$$

where  $0 < \theta < 1$ ,  $\theta$  is independent of  $j$ , and  $C$  depends only on the total degree of  $P(x, y)$ . Thus

$$(3.9) \quad \|T_\infty^{+,b}f\|_{L^p(w)} \leq C \|f\|_{L^p(w)}.$$

Now (3.5) and (3.9) imply that

$$(3.10) \quad \|T_b^+f\|_{L^p(w)} \leq C \|f\|_{L^p(w)},$$

where  $C$  depends not on the the coefficients of  $P(x, y)$ .

*Case 2.*  $|a_{kl}| \neq 1$ . Write  $\lambda = |a_{kl}|^{1/(k+l)}$ , and

$$P(x, y) = \lambda^{-(k+l)} a_{kl} (\lambda x)^k (\lambda y)^l + R(\lambda x/\lambda, \lambda y/\lambda) = Q(\lambda x, \lambda y).$$

Thus

$$\begin{aligned} T^{+,b}f(x) &= \text{p.v.} \int e^{iQ(\lambda x, \lambda y)} b(y-x) K(x-y) f(y) dy \\ &= \text{p.v.} \int e^{iQ(\lambda x, y)} b((y-x)/\lambda) K(\lambda x - y) f(y/\lambda) dy \end{aligned}$$

By Lemma 3.1,  $\|b((\cdot)/\lambda)\|_\infty = \|b\|_\infty$ , so this case goes back to the result in case 1. On account of the estimates for case 1 and case 2 given above, the following can be proved:

$$\|T^{+,b}f\|_{L^p(w)} \leq C \|f\|_{L^p(w)},$$

where  $C$  depends not on the coefficients of  $P(x, y)$ . □

**3.2. Proof of Theorem 1.4.** (a) implies (b): This step is obvious.

(b) implies (c): Write

$$\begin{aligned} T^+f(x) &= \int_x^{1+x} e^{iP(x,y)} K(x-y) f(y) dy + \int_{x+1}^\infty e^{iP(x,y)} K(x-y) f(y) dy \\ &=: T_0^+f(x) + T_\infty^+f(x). \end{aligned}$$

From the method similar to the proof of (3.9),  $T_\infty^+$  is a  $(L^p(w), L^p(w))$  type operator for  $1 < p < \infty$ ,  $w \in A_p^+$ , so is  $T_0^+$ .

Let  $h \in \mathbb{R}$ . Then for  $|x - h| < 1$ ,

$$T_0^+ f(x) \leq T_0^+[f(\cdot)\chi_{I(h,2)}(\cdot)](x),$$

where  $I(x_0, r)$  denotes the interval  $[x_0 - r, x_0 + r]$ . Thus,

$$\begin{aligned} \left( \int_{|x-h|<1} |T_0^+ f(x)|^p w(x) dx \right)^{1/p} &\leq \left( \int_{|x-h|<1} |T_0^+[f(\cdot)\chi_{I(h,2)}(\cdot)](x)|^p w(x) dx \right)^{1/p} \\ &= \|T_0^+ f(\cdot)\chi_{I(h,2)}(\cdot)\|_{L^p(w)} \\ &\leq C \left( \int_{|y-h|<2} |f(y)|^p w(y) dy \right)^{1/p}, \end{aligned}$$

where  $C$  is independent of  $h$ .

Since  $P(x, y) = P(x - h, y - h) + P_0(x, h) + P_1(y, h)$  with  $h \in \mathbb{R}$  and  $P_0, P_1$  are real polynomials defined on  $\mathbb{R}$ , it follows that

$$\begin{aligned} \tilde{T}_0^+ f(x) &\leq p.v \int_x^{x+1} K(x - y) f(y) \chi_{I(h,2)}(y) dy \\ &= p.v e^{-iP_0(x,h)} \int_x^{x+1} e^{iP(x,y)} K(x - y) e^{-iP(x-h,y-h)} e^{-iP_1(y,h)} f(y) \chi_{I(h,2)}(y) dy. \end{aligned}$$

The Taylor's expression of  $e^{-iP(x-h,y-h)}$  is

$$\begin{aligned} e^{-iP(x-h,y-h)} &= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \left[ \sum_{\alpha,\beta} a_{\alpha,\beta} (x-h)^\alpha (y-h)^\beta \right]^m \\ &= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \sum_l C_{m,l} b_{\alpha,\beta,l} (x-h)^{u(\alpha,\beta,l)} (y-h)^{v(\alpha,\beta,l)}. \end{aligned}$$

Therefore, if we set  $|x - h| \leq A < 1$ ,  $|y - h| \leq B < 2$ , the following can be shown:

$$\begin{aligned} &\left( \int_{|x-h|<1} |\tilde{T}_0^+ f(x)|^p w(x) dx \right)^{1/p} \\ &= \left( \int_{|x-h|<1} \left| e^{-iP_0(x,h)} \int_x^{x+1} e^{iP(x,y)} K(x - y) e^{-iP(x-h,y-h)} e^{-iP_1(y,h)} \right. \right. \\ &\quad \left. \left. \times f(y) \chi_{I(h,2)}(y) dy \right|^p w(x) dx \right)^{1/p} \\ &= \left( \int_{|x-h|<1} \left| e^{-iP_0(x,h)} \int_x^{x+1} e^{iP(x,y)} K(x - y) \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \sum_l C_{m,l} b_{\alpha,\beta,l} \right. \right. \\ &\quad \left. \left. \times (x-h)^{u(\alpha,\beta,l)} (y-h)^{v(\alpha,\beta,l)} e^{-iP_1(y,h)} f(y) \chi_{I(h,2)}(y) dy \right|^p w(x) dx \right)^{1/p} \\ &\leq \sum_{m=0}^{\infty} \sum_l \frac{|C_{m,l} b_{\alpha,\beta,l}|}{m!} \left( \int_{|x-h|<1} \left| (x-h)^u \int_x^{x+1} e^{iP(x,y)} K(x - y) \right. \right. \\ &\quad \left. \left. \times (y-h)^v e^{-iP_1(y,h)} f(y) \chi_{I(h,2)}(y) dy \right|^p w(x) dx \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{m=0}^{\infty} \sum_l \frac{|C_{m,l} b_{\alpha,\beta,l}|}{m!} A^u \left( \int_{|x-h|<1} |T_0^+[e^{-iP_1(\cdot,h)} \right. \\
&\quad \left. \times f(\cdot) \chi_{I(h,2)}(\cdot)(\cdot-h)^v](x)|^p w(x) dx \right)^{1/p} \\
&\leq \sum_{m=0}^{\infty} \sum_l \frac{|C_{m,l} b_{\alpha,\beta,l}|}{m!} A^u \left( \int_{|y-h|<2} |f(y)|^p |(y-h)^v|^p w(y) dy \right)^{1/p} \\
&= \sum_{m=0}^{\infty} \sum_l \frac{|C_{m,l} b_{\alpha,\beta,l}|}{m!} A^u B^v \left( \int_{|y-h|<2} |f(y)|^p w(y) dy \right)^{1/p} \\
&= C \sum_{m=0}^{\infty} \frac{1}{m!} \left( \sum_{\alpha,\beta} |a_{\alpha,\beta}| A^\alpha B^\beta \right)^m \left( \int_{|y-h|<2} |f(y)|^p w(y) dy \right)^{\frac{1}{p}} \\
&= C \exp \left( \sum_{\alpha,\beta} |a_{\alpha,\beta}| A^\alpha B^\beta \right) \left( \int_{|y-h|<2} |f(y)|^p w(y) dy \right)^{\frac{1}{p}}.
\end{aligned}$$

Thus

$$\|\tilde{T}_0^+ f\|_{L^p(w)} \leq C \|f\|_{L^p(w)}.$$

(c) implies (a): Set

$$b(r) = \begin{cases} 1, & r \in [0, 1), \\ 0, & r \in [1, +\infty). \end{cases}$$

It is easy to see that  $b(r)$  is a bounded variation function on  $[0, +\infty)$ .

Since the truncated operator  $\tilde{T}_0^+$  is a  $(L^p(w), L^p(w))$  type operator, from Theorem 1.3, the operator  $T_0^+$  is a  $(L^p(w), L^p(w))$  type operator. By the methods similar to the proof of (3.9), the operator  $T_\infty^+$  is a  $(L^p(w), L^p(w))$  type operator. Therefore  $T^+$  is a  $(L^p(w), L^p(w))$  type operator, which implies Theorem 1.4.  $\square$

**3.3. Proof of Theorem 1.5.** The proof of Theorem 1.5 can also be addressed by a double induction on the degree in  $x$  and  $y$  of the polynomial  $P$ . Set

$$P(x, y) = a_{kl} x^k y^l + R(x, y).$$

Since our conclusion is clearly invariant under dialation by the proof of Theorem 1.3, it is suitable to assume that  $|a_{kl}| = 1$ .

If  $k = 0$ , the conclusion holds from the result in [2]. For general  $P(x, y)$ ,

$$\begin{aligned}
T_*^+ f(x) &\leq \sup_{0 < \varepsilon < 1} \left| \int_{x+\varepsilon}^{\infty} e^{iP(x,y)} K(x-y) f(y) dy \right| + \sup_{\varepsilon \geq 1} \left| \int_{x+\varepsilon}^{\infty} e^{iP(x,y)} K(x-y) f(y) dy \right| \\
&\leq \sup_{0 < \varepsilon < 1} \left| \int_{x+\varepsilon}^{x+1} e^{iP(x,y)} K(x-y) f(y) dy \right| + \left| \int_{x+1}^{\infty} e^{iP(x,y)} K(x-y) f(y) dy \right| \\
&\quad + \sup_{\varepsilon \geq 1} \left| \int_{x+\varepsilon}^{\infty} e^{iP(x,y)} K(x-y) f(y) dy \right| \\
&= T_{*,0}^+ f(x) + \left| \int_{x+1}^{\infty} e^{iP(x,y)} K(x-y) f(y) dy \right| + T_{*,\infty}^+ f(x).
\end{aligned}$$

Now, it suffices to prove that  $T_{*,0}^+$  and  $T_{*,\infty}^+$  are  $(L^p(w), L^p(w))$  type operators. By the method similar to prove (3.5),  $T_{*,0}^+$  is  $(L^p(w), L^p(w))$  type operators for  $w \in A_p^+$  and the norm of  $T_{*,0}^+$  depends on the total degree of  $P(x, y)$ , not on the coefficients of  $P(x, y)$ .

For the term  $T_{*,\infty} f(x)$ , there is a  $J \in \mathbb{Z}^+$  such that  $2^{J-1} \leq \varepsilon < 2^J$  allows the following to be shown:

$$\begin{aligned}
T_{*,\infty} f(x) &\leq \sup_{J \in \mathbb{Z}^+} \left| \int_{2^{J-1}}^{2^J} \frac{|f(x-y)|}{|y|} dy \right| + \sup_{J \in \mathbb{Z}^+} \sum_{j=J+1}^{\infty} \left| \int_{x+2^{j-1}}^{x+2^j} e^{iP(x,y)} K(x-y) f(y) dy \right| \\
&\leq M^+ f(x) + \sum_{j=1}^{\infty} \left| \int_{x+2^{j-1}}^{2^j} e^{iP(x,y)} K(x-y) f(y) dy \right|.
\end{aligned}$$

By the boundedness of  $M^+$  and the method similar to prove (3.9), we can derive the following:

$$\|T_{*,\infty}^+ f\|_{L^p(w)} \leq C \|f\|_{L^p(w)}$$

where  $C$  depends on the total degree of  $P(x, y)$ , not on the coefficients of  $P(x, y)$ .  $\square$

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